Stochastic resonance in maps and coupled map lattices

Prashant M. Gade*
Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur, Banglore 560064, India

Renuka Rai and Harjinder Singh
Department of Chemistry, Panjab University, Chandigarh 160014, India
(Received 6 January 1997)

We demonstrate the phenomenon of stochastic resonance (SR) for discrete-time dynamical systems. We investigate various systems that are not necessarily bistable, but do have two well-defined states, switching between which is aided by external noise which can be additive or multiplicative. Thus we find it to be a fairly generic phenomenon. In these systems, we investigate kinetic aspects like hysteresis which reflect the nonlinear and dissipative nature of the response of the system to the external field. We also explore spatially extended systems with additive or parametric noise and find that they differ qualitatively. [S1063-651X(97)12608-3]

PACS number(s): 05.45.+b

I. INTRODUCTION

A seemingly counterintuitive scenario that a weak signal can be enhanced by addition of noise was proposed by Benzi and co-workers [1] in connection with the glaciation cycle of the Earth. Since then stochastic resonance (SR) has been employed in explaining various phenomena (see, e.g., [2,3]) and has been studied extensively experimentally as well as theoretically [4–6]. The major emphasis of the studies has been on the original model by Benzi *et al.* [1] in which the stochastic system in consideration has two stable fixed points in the absence of noise and driving force, though few other models have also received attention [7–10]. The essence of the phenomenon is that even a weak periodic signal which is undetectable in the absence of noise can force a bistable system to switch between its two states, periodically, in the presence of an optimal noise. Often one calculates the power spectrum of the output signal of the system filtered through a two-level filter. The ratio of output power in the frequency of the signal with the background noise, also called a signal to noise ratio (SNR) is a relevent quantifier here. In fact, nonmonotonic behavior of SNR with the noise intensity has become a ''fingerprint'' of this phenomenon.

Additive noise has been the focus of most of the studies. The multiplicative noise, which is not equivalent to additive noise in the presence of a periodic field [11] and thus in principle can show qualitatively different behavior, has been studied in the context of a bistable potential [12]. However, its effect in spatially extended systems, or in systems other than the bistable potential which has two distinct attractors having a possibility of noise induced switching, has not been probed. Multiplicative noise occurs in a variety of physical phenomena [12–14] and is certainly important. Recently alternative quantifiers are suggested to study SR. Most notable among them is the residence time distribution function (RTDF). Unlike SNR, areas under successive peaks of RTDF show nonmonotonicity, both as a function of frequency or noise intensity [15]. Also of importance are the kinetic aspects of the phenomenon which reflect the way a system responds to the signal. These aspects have been mainly investigated by measuring the phase shift between the signal and the response [16,17]. Hysteresis which reflects phase shift and also the extent of losses in a periodically driven system is an equally important quantifier [18–20]. However, studies of the usual hysteresis behavior in the presence of noise have hardly been attempted. One notable exception has been a numerical work by Mahato and Shenoy [18]. However, they define the hysteresis loop in a different way than is usually done. Thus, as they state, we cannot expect their results to be carried over to the usual case even qualitatively. Recently there have been few interesting studies on SR in spatially extended systems theoreticaly [21–24], as well as experimentally [25]. The reasons for the recent surge of interest in analysis of spatiotemporal systems are not far to seek. These systems are important from the point of view of potential applications that range from coupled nonlinear devices and signal processing to neurophysiology and merit further attention.

We feel that SR is a generic feature of two-state systems, neither state of which needs to be a stable fixed point. Any system with two well-defined and well-separated states switching between which can be aided by noise, can possibly show stochastic resonance in the presence of a weak signal. We will illustrate this with systems switching between two chaotic attractors, a chaotic attractor and a fixed point, and will also present the standard model of two stable fixed points. It is easy and computationally inexpensive to construct such cases using discrete-time systems like maps and we will be using maps for demonstration. We have used both additive and multiplicative noise in our simulations. In many physical and chemical systems noise is generated internally (see, e.g., [13]) and such systems have been observed to show stochastic resonance. As we have pointed out, the multiplicative noise in the presence of a periodic force is not equivalent to additive noise coupled with a periodic signal. Interestingly, in the case of a single map, both show a similar qualitative behavior as far as SR is concerned, i.e., in both cases SNR shows nonmonotonic behavior of response as a function of noise and has a peak at some value of noise.

*Electronic address: prasha@jnc.iisc.ernet.in

We have investigated residence time distribution function as a function of frequency and noise intensity in this system. We discover that the behavior is analogous to the one found in an archetypical example of a bistable potential [15].

In this dissipative and nonlinear system, the response to the external field is likely to be delayed and nonlinear. The simplest way to gauge it is to study hysteresis in these systems which will reflect both losses as well as delay in the response [18]. This will be information additional to the one given by a simple signal to noise ratio, e.g., SNR does not reflect the phase shift in the response.

Finally we investigate spatially extended systems where local dynamics is governed by these maps. Such systems, popularly known as coupled map lattices (CML's) [26] have gained considerable attention in recent times due to their computational simplicity and ability to reproduce qualitative features in various phenomena. To name a few interesting applications one can point out the modeling of phase ordering dynamics [27], spatiotemporal intemittency [28], spiral waves [29], etc. However, we are not aware of any attempt to observe SR in these systems. Here we see a qualitative difference between the systems subjected to an additive noise and parametric noise.

In Secs. II and III we will define our models and present their analysis. In Sec. IV we discuss the residence time distribution function and how it is affected by noise intensity and frequency. In Sec. V we will discuss the hysteresis and effect of noise. In Sec. VI we will define the spatially extended systems in a way popularly known as a coupled map lattice and discuss the results in it. Finally in Sec. VII we conclude and discuss questions that we are interested in.

II. THE MODELS

Let us consider the following maps in absence of any periodic or noisy drive:

$$f_S(x)=S\tanh(x),\quad x\in(-\infty,\infty)S>0, \tag{1}$$

$$h_a(x)=\exp(a)x|_{\mathrm{mod}\,1},\quad x\in(-1,1), \tag{2}$$

$$g_r(x)=\begin{cases} rx|_{\mathrm{mod}\,1}, & x\in\left(0,\frac{1}{2}\right], \\ r(1-x)|_{\mathrm{mod}\,1}, & x\in\left(\frac{1}{2},1\right], \end{cases}$$

$$g_r(x)=-h_r(-x)x\in(-1,0). \tag{3}$$

(a) The map $f_S(x)$ which has a Hamiltonian symmetry $[f_S(-x)=-f_S(x)]$ and two stable fixed points symmetric around $x=0$ is clearly analogous to the original model of the bistable potential [1]. In this case also, the basin of attraction of the positive fixed point x_S^* is $(0,\infty)$ while that of the negative fixed point that is symmetrically placed at $-x_S^*$ is $(-\infty,0)$. Thus the system has two stable fixed point attractors any of which is reached depending on initial conditions [see Fig. 1(a)] [30].

(b) The map $h_a(x)$ shows chaotic behavior if $a>0$ or has a stable fixed point 0 as an attractor if $a<0$. Thus the map has a chaotic or fixed point attractor depending on the value of a. [Figure 1(c) shows the map $h_p(x)$ and $h_q(x)$ for $p>0$ and $q<0$.]

(c) The map $g_r^{(x)}$ also has the same symmetry as map $f_S(x)$. It has a fixed point attractor at $x=0$ for $|r|<1$. However for $|r|>1$, it has an interesting behavior. In this range, it shows two chaotic attractors symmetric around zero separated by $2a_0$ [see Fig. 1(b)]. The map is such that for any initial condition $x_0\in[0,1]$, $g_r^T(x_0)\in[0,1]$ for all times T. In fact, for $1<r<2$ the attractor on the positive side does not span an entire unit interval but is in the interval $[a_0,a_1]=[g_r^2(\frac{1}{2}),g_r(\frac{1}{2})]$ for $x\in(0,1)$, while for $x\in(-1,0)$, the values asymptotically span interval $[-a_1,-a_0]$. (For $r\geqslant 2$, $a_0=0$, $a_1=1$.)

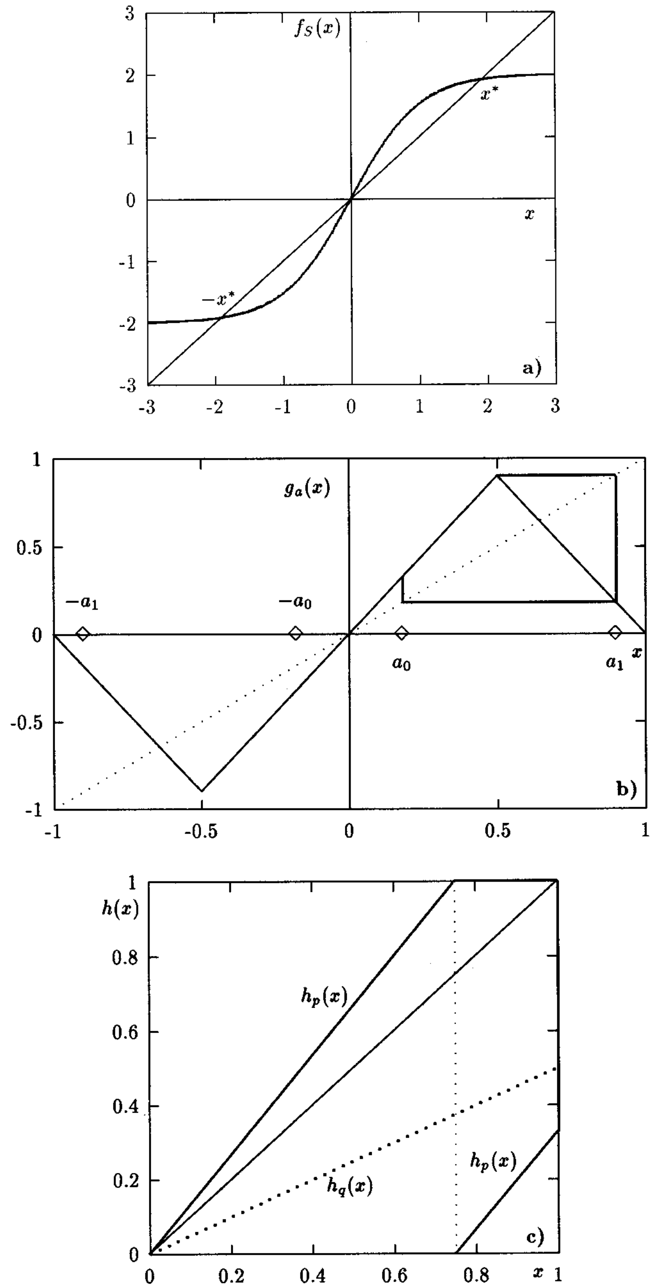

FIG. 1. The maps f_S, g_a and h_r are shown in as defined in Eqs. (1), (2) and (3) and are depicted in (a), (b), and (c), respectively.

III. SR IN MAPS

The above maps have the desired property of having two different attractors between which the system can switch when aided by noise. We investigate the following systems:

$$\text{(a)}\quad x(t+1)=f_S(x(t))+z\cos(2\pi\omega t)+\eta_t, \tag{4}$$

$$\text{(a1)}\quad x(t+1)=f_{S+\eta_t}(x(t))+z\cos(2\pi\omega t), \tag{5}$$

$$\text{(b)}\quad x(t+1)=[h_{a+\eta_t+z\cos(2\pi\omega t)}(x(t)-p_0)]+p_0|_{\text{mod }1}, \tag{6}$$

$$\text{(c)}\quad x(t+1)=[g_r(x(t))+z\cos(2\pi\omega t)+\eta_t]|_{\text{mod }1}, \tag{7}$$

$$\text{(c1)}\quad x(t+1)=[g_{r+\eta_t}(x(t))+z\cos(2\pi\omega t)]|_{\text{mod }1}, \tag{8}$$

where η_t is a δ correlated random number with variance D. Except in the case of Eq. (4), where η_t has a Gaussian distribution, we have a uniform distribution for η_t. Due to symmetry, the maps $f_S(x)$ and $g_r(x)$ have symmetric attractors on the positive and negative sides. Since we are interested in interstate switching we neglect the intrastate fluctuations in the cases of Eqs. (4), (5), (7) and (8) while analyzing the output. The output is analyzed in the usual manner, i.e., one takes the Fourier transform of the time series thus generated and averages the power over various phases and also initial conditions. The SNR is defined as the ratio of the intensity of a δ spike in the power spectrum at the frequency $\Omega=2\pi\omega$ to the height of the smooth fluctuational background $Q^0(\Omega)$ at the same frequency Ω then

$$\text{SNR}=\log_{10}\frac{(\text{total power in the frequency } \Omega)}{Q^0(\Omega)}. \tag{9}$$

Variations in this definition do not change the results qualitatively. Our computations are at $2\times 3^2/\omega$ grid points and we have used the Bartlett function for windowing. (For a discussion of technical aspects, see Ref. [4].)

(a) This is the simplest system which mimics the well-studied bistable potential model [1] of Benzi and co-workers. Here the system toggles between the positive and the negative fixed points of the map f_S, (x_S^* and $-x_S^*$). The system is defined over the entire range $(-\infty,\infty)$. We apply Gaussian noise. It is seen that as the noise intensity increases the spectral strength of the signal also increases, but this happens at the expense of noise, thus increasing the signal to noise ratio. This happens since, when the signal is at its peak, the little noise aids the system to flip from the basin of attraction of one fixed point to other. For large noise, the flips can occur almost all the time, the regularity is reduced and SNR decreases again. This behavior is shown in Fig. 2(a) which shows SNR as a function of noise intensity D for the system defined by Eq. (4).

(a1) In this map we also have the possibility of parametric noise as in Eq. (5). The value of noise changes the position and the stability of the fixed point. For large enough noise the fixed point can come arbitrarily close to zero, which coupled with the periodic signal can cause flips which can be very regular at optimal noise level. Thus SNR shows as standard SR behavior as a function of D [see Fig. 2(b)].

(b) Now we explore the possibility of competition between a fixed point attractor and a chaotic attractor switching between which is aided by noise. [We have added a small constant p_0 in Eq. (6) unlike Eq. (2) for numerical reasons. The position of the fixed point now changes to p_0. For $p_0=0$, the trajectory that comes close to zero within numerical precision will stay there. This change does alter the description of the map qualitatively.] At the minimum value of the drive, it is likely that $a_t=a+\eta_t+z\cos(2\pi\omega t)<0$ and the system will be attracted to the fixed point. While small noise will aid this repetitive attraction towards the fixed point, very large noise is likely to reduce it. As a result, we see a nonmonotonic response of SNR to noise intensity. In some sense, this system mimics excitable dynamics, where SR has been observed [10] (though the excited state in this case is not a chaotic state). Figure 2(c) shows SNR as a function of noise intensity D for this system.

This system is like a random walk when viewed on a logarithmic scale. Neglecting the modulo factor, the variable value at time $n+1$ goes as $\ln(x_{n+1})=a_n+a_{n-1}+\cdots+a_0+\ln(x_0)$ where $a_t=a+\eta_t+z\cos(2\pi\omega t)$. Thus it is like a random walk with initial position $\ln(x_0)$ and dispacement a_i at ith time step. The value of x_i is bounded from above by unity due to the modulo condition and x_i does not tend to zero asymptotically since $a>0$. Due to $a>0$, this is a case of a biased one-dimensional (1D) random walk bounded from above. Thus this system is comparable with the stochastic resonance seen in random walk [31]. Figure 3 shows a schematic diagram for the above description.

(c) This is another interesting but unexplored possibility. Here we have two chaotic attractors switching between which is aided by noise and a periodic signal. We would like to point out that apart from the nature of the attractors, this system is very similar to the system defined by Eq. (4) as far as the dynamics of interstate switching is concerned. Let us consider the system defined in Eq. (7). As noted above, the two attractors are well separated in the absence of noise and periodic signal and the system stays in either of them depending on initial conditions. However, in the presence of noise, the system can ''leak'' out of the attractor. This ''leaking,'' i.e., switching from positive attractor to negative and vice versa is likely to occur at the most opportune times, i.e., at the minimum and the maximum of the signal. Thus one may expect stochastic resonance here which is indeed the case [see Fig. 2(d)].

(c1) Here one could have a parametric noise as an alternative to additive noise. The parametric noise can change the value of a_0 which controls the distance between two attractors thus aiding the switching. Figure 2(e) shows SNR as a function of noise intensity D in these systems. One can see a clear nonmonotonicity in response.

IV. RESIDENCE TIME DISTRIBUTION FUNCTION

Of late, quantifiers, other than SNR have been proposed in the context of SR. To name a few, the signal amplitude and residence time distribution function (RTDF) have been discussed in the literature. In our opinion, SNR is a better quantifier than the signal amplitude itself from the signal detection point of view. However, the residence time distribution function is an important quantity from the following

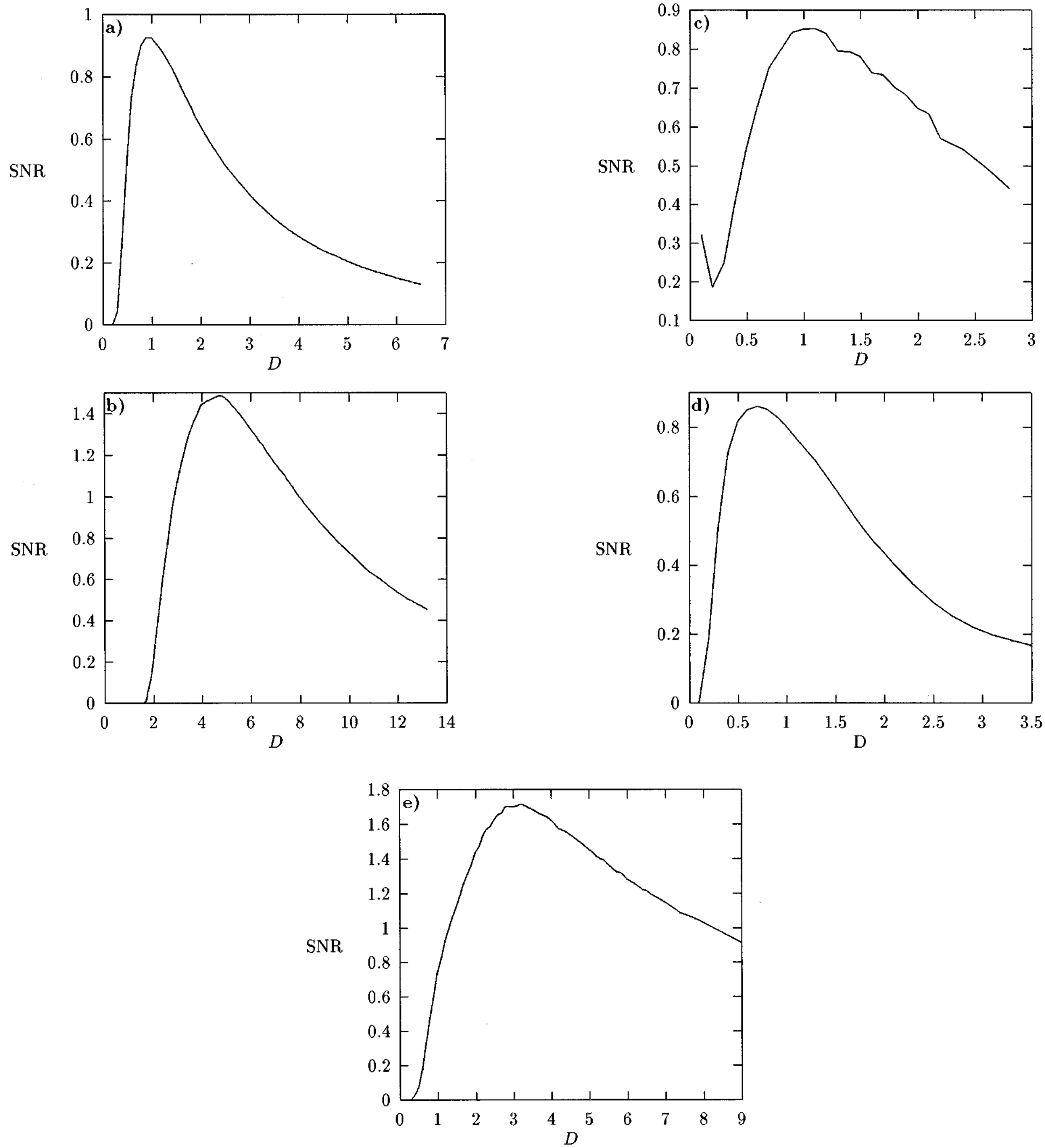

FIG. 2. The SNR as a function of D for systems defined by Eqs. (4)–(8) are shown in (a)–(e), respectively. $\omega=1/8$, $S=2$, $z=0.5$ for (a) and (b); $\omega=1/8$, $r=1.4$, $z=0.12$ for (d) and (e); and $\omega=1/32$, $p_0=0.01$, $a=0.1$, $z=0.1$ for (c). Noise is Gaussian for the case of the unbounded map defined by Eq. (4) while it is uniform on a unit interval in other cases.

point of view. The intuitive argument given for SR is that when the Kramer's rate, which is a time scale induced by the stochastic process, matches with the periodicity of the forcing term, the output signal shoots up. Thus changing either time scale, i.e., Kramer's rate which can be altered by noise, or the periodicity of the forcing term, one should observe the SR. However, though SNR shows nonmonotonicity as a function of noise intensity, it is a steadily decreasing function of the forcing frequency unlike the ordinary resonance where resonance can be achieved by changing either time scale [4]. So the question is wheher SNR is the right quantity to look for. One more problem pointed out is that the maximum of SNR does not occur when the Kramer's rate exactly matches the forcing frequency. In fact, like several others [5,32], Fox [33] pointed out that the peak in SNR has nothing to do with the matching of Kramer's rate to the signal frequency and suggested the term ''noise induced signal to noise ratio enhancement (SNRE)'' which, though a clumsier term than SR, is probably a more correct description of the phenomenon in his opinion.

In this context RTDF has been introduced as an alternative quantifier. If t_i denotes the times at which successive

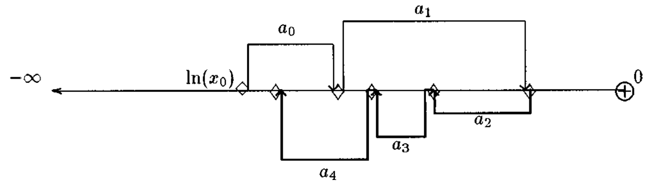

FIG. 3. This figure shows the schematic diagram of how the evolution under Eq. (6) resembles a 1D random walk on a logarithmic scale with a boundary condition at $\ln(x)=0$ coming due to the modulo condition on the right-hand side while the evolution is unbounded on the left-hand side.

switchings occur, normalized distribution $N(t)$ of the quantities $T(i)=t_i-t_{i-1}$ is called RTDF. The observation is that the RTDF shows peaks centered at $T_n=(n-\frac{1}{2})T_0$, $T_0=1/\omega$ on an exponentially decaying background. The quantity $n=1,2,\ldots$ is a positive integer and the peaks at succesive T_i's are also exponentially decaying. [This behavior is observed by applying this definition to a system defined by Eq. (4) also. See Fig. 4 which shows RTDF for three different frequencies.] This quantifier is attractive since it has been shown by Gammatoni *et al.* that if one computes the area under succesive peaks it shows a nonmonotonicity as a function of both noise intensity as well as forcing frequency [15]. Thus the intuitive picture of SR is recovered. In analogy with [15], we defined areas under different peaks P_n as

$$P_n=\sum_{T_n-T_0/4}^{T_n+T_0/4} N(T)dt. \tag{10}$$

Except for very large frequencies, i.e., for very small periodicities $(T_0\leqslant 4)$, this definition works well to check the peaks. The behavior of areas under different peaks for the system defined by Eq. (4) is plotted in Fig. 5(a) as a function of frequency ω. It shows a clear nonmonotonicity as a function

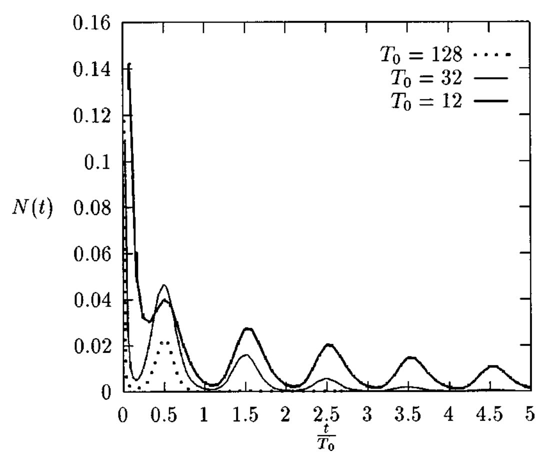

FIG. 4. (a) Hysteresis curve for $D=0.2$, $D=1$, $D=1.4$, and $D=5.2$ for the map defined by Eq. (4) for $S=2$, $z=0.5$, and $T=1/\omega=32$. One can clearly see that for low D maxima and minima of magnetization are not in tune with the drive. (b) The area of hysteresis loop as a function of noise intensity D in this case. One can clearly see a decrease at larger noise values. Residence time distribution functionn (RTDF) $N(t)$ as a functiion of scaled time t/T_0 is shown in the figure for various frequencies, i.e., various values of T_0. Parameters are $S=2$ and $z=0.5$.

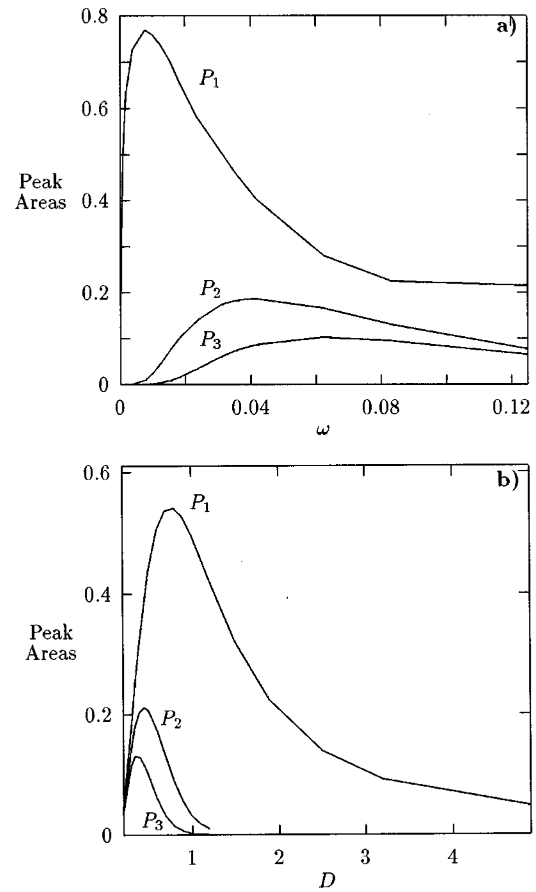

FIG. 5. (a) The area under the first, second, and third peaks of RTDF [P_1, P_2, and P_3, as defined in Eq. (10)] is shown as a function of driving frequency at a fixed value of noise intensity $D=0.6$. (b) The area under the first, second, and third peaks of RTDF (P_1, P_2, and P_3) is shown as a function of noise intensity at a fixed driving frequency $(\omega=1/32)$. (Parameters are $S=2$ and $z=0.5$.)

of freqency. This is also clear from Fig. 4 where RTDF is shown for different frequencies. The height of the peaks first grows and then decays. The dependence of the area under different peaks as a function of noise intensity at a given frequency for the same system is shown in Fig. 5(b) which is also clearly nonmonotonic. It is interesting that the the behavior of this system even in quantifiers other than SNR is very analogous to the one seen in continuous time systems.

V. HYSTERESIS

Now we discuss the kinetic aspects of this phenomenon. Hysteresis is a kinetic phenomenon which is the signature of the response of the system to an external field sweep. In general, due to the losses, the system is not able to follow the external signal exactly. There is an accumulated strain after which it responds to the signal. The quantity that reflects these losses is the area of the hysteresis loop. The most familiar example is the behavior of magnetization M as a function of alternating external magnetic field H. We have a two-state system in our examples and we are analyzing the signal filtered through two-state filter. This makes it easy to define an analogue of magnetization. We define

$$m(t)=\frac{1}{N}\sum_{j=1}^{N} \operatorname{sgn}[x(jT+t+1)],$$

where T is the period of the applied periodic force and $t=1,2,\ldots,T/2$. It is clear that $m(t)$ is just the difference between the number of times the value of the variable x is greater than zero and that it is less than zero, at times $t+1$ modulo T where $1 \leqslant t \leqslant T/2$. We normalize $m(t)$ properly so that $r(t)=m(t)/M$ is confined between 1 and -1. [M is the maximum value that $m(t)$ takes. By symmetry, we would expect it to be the same on the positive and negative sides.] This gives one of the branches of the hysteresis loop. The other branch can be constructed by symmetry or computed by

$$m(T/2-t)=\frac{1}{N}\sum_{j=1}^{N} \operatorname{sgn}[x(jT+T/2+t+1)]$$

for $t=1,2,\ldots,T/2$ and $r(t)=m(t)/M$. Here one can have two extreme cases. If the response is exactly in tune with the field, the magnetization will be zero at a zero value of the periodic force (no remnant magnetization) and the hysteresis area will be zero. On the contrary, if the response is so late that only at the end of the half-cycle, the flips start occuring, the loop will have maximum area. Thus the more delayed the response is, the higher is the area of the hysteresis loop and hence is the popular notion of hysteresis area giving indication of losses in the system. One would *a priori* think that as the noise intensity D increases, which is an equivalent of increasing temperature, the hopping between the different states will be faciliated and thus the area of the loop will decrease at larger noise values. This is exactly what our observation is. We have plotted the area of hysteresis loop for the system defined by Eq. (4) in Fig. 6(b). Here, we see a clear decrease in hysteresis loop area with increase in noise intensity. As noise increases, the losses are reduced since the system will not stay in the metastable state for long. A sudden fluctuation will force it to respond to the signal and there will be little memory or remnant magnetization in the system. Apart from the area of the hysteresis loop, the shape of the hysteresis is also an interesting object to investigate. Though for a noise higher than some critical value, the maxima and minima in magnetization start occurring at the maxima and minima of the field, for smaller noise they occur at different times. [See Fig. 6(a) where we have plotted the hysteresis loop for four different values of noise intensity.] We can see that the response is delayed from the field by a finite phase shift. This phase shift reduces with increasing noise [34]. The results indicate that in the limit of small noise the phase shift is of the order of a quarter of the total period, $-\pi/2$ which is expected [16] for a two-state analysis that does not take in account intrawell fluctuations.

VI. SR IN COUPLED MAP LATTICES

Let us discuss cooperative phenomena possible in the spatially extended versions of this system. The spatially discretized periodically forced time dependent Ginzburg-Landau equation [23], as well as a one-dimensional array of coupled bistable oscillators [21], have been studied before.

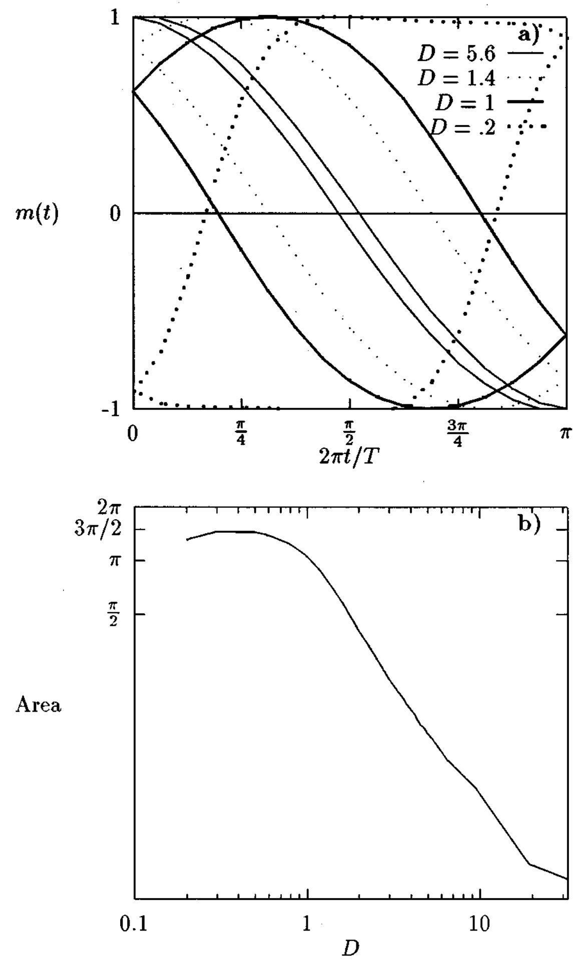

FIG. 6. (a) Hysteresis curve for $D=0.2$, $D=1$, $D=1.4$, and $D=5.2$ for the map defined by Eq. (4) for $S=2$, $z=0.5$, and $T=1/\omega=32$. One can clearly see that for low D maxima and minima of magnetization are not in tune with the drive. (b) Area of hysteresis loop as a function of noise intensity D in this case. One can clearly see a decrease at larger noise values.

The major result in the work by Lindner and co-workers [21] has been that the largest value of SNR is higher for a given oscillator of the coupled system as compared to the uncoupled one. However, the maximum of SNR does not occur at the same value of noise intensity. Not only is the best value of SNR higher for the coupled case, the value of SNR at a given value of noise intesity D is better for the coupled system as compared to the uncoupled one. A simple interpretation that can be offered for the above observation is that even when an oscillator misses the interstate switching, the nearby oscillators may not, thus forcing the individual oscillator to switch. This cooperative behavior should induce enhanced regularity in the switching of the oscillators and increase SNR for a given oscillator. The simple interpretation which can be true for any interstate switching mechanism should also work for the systems we are studying.

We define the following spatially extended system. We will follow an evolution scheme popularly known as a coupled map lattice. Let us consider a linear array of length N. At each lattice point i ($i=1,\ldots,N$) we attach a variable $x_i(t)$ at time t. The time evolution of $x_i(t)$ is described by

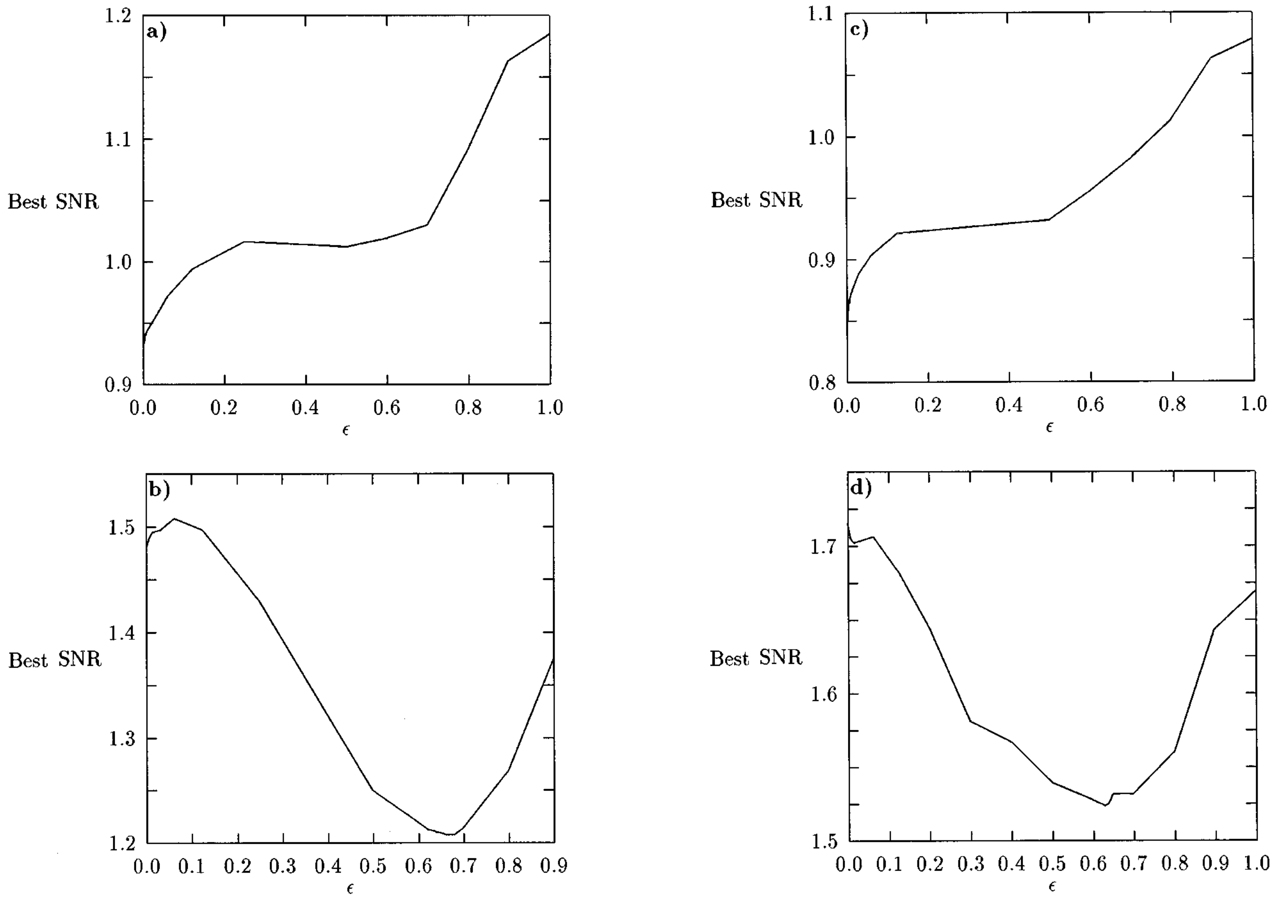

FIG. 7. Best value of SNR as a function of coupling ϵ for maps defined by Eqs. (4), (5), (7), and (8) are shown in (a), (b), (c), and (d). It is clear that while for additive noise the best SNR is enhanced, it is not so for parametric noise.

$$x_i(t+1)=(1-\epsilon)F(x_i(t))+\frac{\epsilon}{2}[F(x_{i-1}(t))+F(x_{i+1}(t))] \tag{11}$$

for $2\leqslant i\leqslant N-1$ and

$$x_1(t+1)=(1-\epsilon)F(x_1(t))+\epsilon F(x_2(t)),$$

$$x_N(t+1)=(1-\epsilon)F(x_N(t))+\epsilon F(x_{N-1}(t)), \tag{12}$$

where F denotes some time evolving map. Of course, we will be interested, in particular, in single maps that show SR [e.g., Eq. (4)].

We have used the maps defined by Eqs. (4), (5), (7) and (8) as a function F in the above equation and have made a detailed study of SNR at various values of coupling ϵ and noise intensity D for $N=8$. Following Lindner *et al.* [21] we have looked at the response of the middle oscillator. Figures 7(a) and 7(b) show the best value of SNR as a function of coupling for maps (4) and (5), i.e., the tanh map with additive and parametric noise while Figs. 7(c) and 7(d) show the same for maps defined by Eq. (7) and Eq. (8), i.e., a chaotic map with additive and multiplicative noise, respectively. We have the following observations. (a) In all these cases, the SNR of the middle oscillator as a function of noise for a given value of coupling is nonmonotonic and shows a peak at some optimal value as in the single map case. The optimal value of noise need not be the same as one for an uncoupled case. (b) If the single system is perturbed with additive noise the coupling between such systems always enhances the SNR, i.e., the best SNR for the coupled system is better than the one for a single oscillator case. (c) If the single system is perturbed by parametric noise, the coupling between such systems does not enhance SNR much. In fact for a large value of coupling, there is a clear decrease in SNR.

While observations (a) and (b) are in tune with the studies by Lindner *et al.* [21] in coupled bistable systems, the case with parametric noise has not been studied before.

We feel that the reason for this qualitative difference is following. When two systems with different parameters are coupled, i.e., a system with a high Kramer's rate is coupled to one with a low Kramer's rate, the Kramer's rate for the coupled system is like that of the slower system [35]. For higher coupling, this effect is more pronounced. Thus, instead of an induced switching, one could have a slowed down switching in the presence of coupling. In this context, we would like to point out a rather curious outcome of our numerical investigation that the best SNR has the least value around $\epsilon=2/3$ in the case of parametric noise. This is the value at which all the maps in the neighborhood have equal weight in Eq. (11) $(1-\epsilon=\epsilon/2=1/3)$.

VII. DISCUSSION

Due to the computational simplicity of the system defined above, and its ability to produce key features, the system defined above holds promise for carrying out work in various directions with relative ease in the future. Here we would like to point out that using a coupled map like formulation, Oono and Puri have been able to get a *quantitative* agreement in the modeling phase ordering dynamics of Ising-type systems [27]. Miller and Huse have also found that chaotic coupled maps with Hamiltonian symmetry show a phase transition with static and dynamic critical exponents consistent with the Ising class [36]. On the other hand, the Ising system has been reported to show SR in 1D and 2D [37,38]. As we have pointed out, recently there have been studies on coupled bistable oscillators in 1D which show SR [21]. Although one does not expect all the detailed behavior in one model to carry over to the other, all these things do point towards a broader universality between coupled bistable oscillators, coupled maps, and Ising systems. The investigations on these lines should be useful in understanding SR in spatially extended systems which are relatively unexplored but clearly important from various points of view. Given the wide variety of physical situations that the Ising model is able to simulate, this similarity should not come as a surprise. In fact, the similarity between coupled bistable oscillators and Ising-type systems has been pointed out before [39]. While studies of the Ising system itself will be useful in analytic investigations, investigations in the coupled-map-type systems will be computationally more efficient. Studies in globally coupled maps [24] and detection of noise induced transitions in maps [40] could be carried out in these systems. The behavior of SNR as a function of coupling ϵ and number of maps N could be studied further and these investigations are in progress. One could also investigate SR in a two-dimensional coupled map lattice since higher dimensional spatially extended systems are not investigated. One more important question that needs to be addressed is the effect of multiplicative noise and disorder in SR in spatially extended systems.

ACKNOWLEDGMENTS

The authors have enjoyed discussions with Professor N. Kumar (RRI). R.R. would like to thank UGC for financial support and RRI for hospitality while H.S. would like to thank the Indian Academy of Science for support for the visit to JNCASR where the work was carried out.

[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); R. Benzi *et al.*, Tellus **34**, 10 (1982).
[2] K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995); A. R. Bulsara and L. Gammatoni, Phys. Today **49**(3), 39 (1996).
[3] See, for example, Proceedings of the NATO ARW Stochastic Resonance in Physics and Biology, edited by F. Moss, A. Bulsara, and M. F. Shlesinger [J. Stat. Phys. **70**, 1 (1993)].
[4] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1988).
[5] S. Fauve and F. Heslot, Phys. Lett. **97A**, 5 (1983).
[6] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).
[7] Recently nondynamical models of this phenomenon have been proposed. See Z. Gingl, L. B. Kiss, and F. Moss, Europhys. Lett. **29**, 191 (1995); L. Gammatoni, Phys. Rev. E **52**, 4691 (1995); J. J. Brey, J. Casadopascual, and B. Sanchez, *ibid.* **52**, 6071 (1995): S. M. Bezrukov and I. Vodyanoy, Nature (London) **385**, 319 (1997).
[8] There has been a study on SR in a monostable well [9]. There have also been studies on systems with excitable dynamics [10]. The two states of this system are a stable fixed point and an excited state. The other state is not a stationary state and the system returns back to the fixed point after a certain refractory period.
[9] N. G. Stocks *et al.*, J. Phys. A **26**, L385 (1993).
[10] K. Wiesenfeld *et al.*, Phys. Rev. Lett. **72**, 2125 (1994).
[11] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1984).
[12] L. Gammatoni *et al.* Phys. Rev. E **49**, 4878 (1994).
[13] D. S. Leonard and L. E. Reichl, Phys. Rev. E **49**, 1734 (1994).
[14] R. Graham and A. Schenzle, Phys. Rev. A **25**, 1731 (1982).
[15] L. Gammatoni, F. Marchesoni, and S. Santicci, Phys. Rev. Lett. **74**, 1052 (1995).
[16] L. Gammatoni *et al.*, Phys. Lett. A **158**, 449 (1991); L. Gammatoni and F. Marchesoni, Phys. Rev. Lett. **70**, 874 (1993).
[17] M. Dykman *et al.* Phys. Rev. Lett. **70**, 874 (1993); P. Jung and P. Hanggi, Z. Phys. B **90**, 255 (1993); M. Morillo and J. Gómez-Ordóñez, Phys. Rev. Lett. **71**, 9 (1993).
[18] M. C. Mahato and S. R. Shenoy, Phys. Rev. E **50**, 2503 (1994).
[19] M. Rao, H. R. Krishnamurthy, and R. Pandit, Phys. Rev. B **42**, 856 (1990); J. Phys. Condens. Matter **1**, 9061 (1991).
[20] D. Dhar and P. B. Thomas, J. Phys. A **25**, 4967 (1992).
[21] J. F. Lindner *et al.*, Phys. Rev. Lett. **75**, 3 (1995); J. F. Lindner *et al.*, Phys. Rev. E **53**, 2081 (1996); see also F. Marchesoni, L. Gammatoni, and A. R. Bulsara, Phys. Rev. Lett. **76**, 2609 (1996).
[22] H. S. Wio, Phys. Rev. E **54**, R3075 (1996).
[23] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **18**, 2239 (1985).
[24] P. Jung, U. Behn, E. Pantazelou, and F. Moss, Phys. Rev. A **46**, 1709 (1992).
[25] M. Locher, G. A. Johnson, and E. R. Hunt, Phys. Rev. Lett. **77**, 4698 (1996).
[26] See, e.g., J. P. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao-Bai-lin (World Scientific, Singapore, 1987).
[27] Y. Oono and S. Puri, Phys. Rev. Lett. **58**, 836 (1987); Phys. Rev. A **38**, 434 (1988); **38**, 1542 (1988).
[28] H. Chate and P. Mannevile, Physica D **32**, 409 (1988).
[29] D. Barkley, in *Nonlinear Structures in Dynamical Systems*, edited by Lui Lam and H. C. Morris (Springer-Verlag, New York, 1990).
[30] We would like to point out that this map has been used to simulate time dependent Ginzburg-Landau equations (TDGL's) by Puri and Oono [27].
[31] D. S. Leonard, Phys. Rev. A **46**, 6742 (1992).

[32] Hu Gang, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).
[33] R. F. Fox, Phys. Rev. A **39**, 4148 (1989).
[34] It is difficult to get reliable statistics at lower noise intensities, since there are fewer flips.
[35] A. Neiman, Phys. Rev. E **49**, 3484 (1994).
[36] J. Miller and D. A. Huse, Phys. Rev. E **48**, 2528 (1993).
[37] J. J. Brey and A. Pandos, Phys. Lett. A **216**, 240 (1996).
[38] Z. Néda, Phys. Rev. E **51**, 5315 (1995).
[39] L. Kiss *et al.*, J. Stat. Phys. **70**, 451 (1993); M. Inchiosa and A. R. Bulsara, Phys. Rev. E **52**, 327 (1995); P. Jung and G. Mayer-Kress, Phys. Rev. Lett. **74**, 2130 (1995).
[40] H. Gang, H. Haken, and X. Fagen, Phys. Rev. Lett. **77**, 1925 (1996).